Magnetic monopoles in $U(1)_4$ Lattice Gauge Theory with Wilson action

V. Cirigliano [a,b] and G. Paffuti [a,c]
[a] *Dipartimento di Fisica dell'Università and I.N.F.N., P.zza Torricelli 2, I-56100 Pisa, Italy*
[b] E-mail: ciriglia@ipifidpt.difi.unipi.it
[c] E-mail: paffuti@ipifidpt.difi.unipi.it

We construct the Euclidean Green functions for the soliton (magnetic monopole) field in the $U(1)_4$ Lattice Gauge Theory with Wilson action. We show that in the strong coupling regime there is monopole condensation while in the QED phase the physical Hilbert space splits into orthogonal soliton sectors labelled by integer magnetic charge.

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Contents

I. INTRODUCTION

In this paper we apply the methods introduced in [1] to the construction of soliton (magnetic monopole) sectors for the $U(1)$ Lattice Gauge Theory with Wilson action [2]. The soliton quantization for the $U(1)$ Lattice Gauge Theory with Villain action [3], as well as for a large class of models, has been carried out in [1,4] by constructing the Euclidean Green functions of soliton fields as expectation values of suitable disorder operators. These operators are obtained by coupling the theory to a generalized external gauge field in an hyper-gauge invariant way, as we shall briefly recall below. In the statistical approach this procedure corresponds to the introduction of open-ended line defects. In the case of the $U(1)_4$ gauge theories such line defects are just magnetic loops carrying a defect (topological) charge. Opening up a loop one introduces magnetic monopoles at the endpoints of the current line. An Osterwalder-Schrader (O.S.) reconstruction theorem [5,6] applied to the disorder fields correlation functions permits the identification of vacuum expectation values of the soliton field, which can be considered as a charged operator creating magnetic monopoles [7,8].

We have focused our attention in particular on the vacuum expectation value of the soliton field, which is defined by a limiting procedure starting from the two point function:

$$S_1 = \lim_{|x|\to\infty} S_2(0, x)\ .$$

The result of our work is that S_1 is a good disorder parameter for the phase transition occurring in the model [9]. In fact we show that S_1 is bounded away from zero in the strong coupling regime ($\beta \ll 1$) while is vanishing in the

QED phase. The proof of this statement makes use of two different techniques. The strong coupling phase is analysed by means of a convergent Mayer expansion, applied to a polymer system obtained by the dual representation of the model [10]. Besides, the clustering property in the QED phase is proved using an adapted version of the expansion in renormalized monopole loops originally given by Fröhlich and Spencer [11,12]. The dual representation of the Wilson model has a measure given by products of modified Bessel functions: the extimates have been done applying suitable bounds on modified Bessel functions for $\beta \ll 1$, while using for β large an interpolation to Bessel functions given in [11].

The paper is organised as follows: in this section we shortly define the disorder fields and correlation functions for the $U(1)$ Wilson model and describe their connection with magnetic monopoles. Then we enunciate the reconstruction theorem for the $U(1)$ Wilson model. In section II we'll map the model into a polymer system in order to prove that S_1 is nonvanishing in the strong coupling regime and thus the lattice solitons "condense" in the vacuum sector. In section III we'll give the expansion in renormalized monopole loops in order to show that $S_1 = 0$ in the weak coupling regime: the Hilbert space of the reconstructed Lattice Quantum Field Theory splits into orthogonal sectors labelled by the magnetic charge. Finally, in section IV we give some concluding remarks.

A. Disorder fields and magnetic monopoles

In what follows we consider all fields as defined on a finite lattice $\Lambda \subset Z^4$; all estimates needed to proof our statements are uniform in $|\Lambda|$ and therefore extend to the thermodynamic limit ($\Lambda \to Z^4$). The partition function for the $U(1)$ gauge model is

$$Z = \int \mathcal{D}\,\theta \prod_{p\subset\Lambda} \varphi_\beta\left(d\theta_p\right) \,, \tag{1}$$

where $\mathcal{D}\theta$ is the product measure on the 1-form θ valued in $[-\pi, \pi)$, $d\theta$ is the field strength defined on the plaquette p and (for Wilson action)

$$\varphi_\beta\left(d\theta\right) = e^{\beta\,\cos(d\theta)} \quad . \tag{2}$$

To define a disorder operator we can consider a modified partition function in which an external hyper-gauge field strength X is coupled to the dynamical variables:

$$Z(X) = \int \mathcal{D}\,\theta \prod_{p\subset\Lambda} \varphi_\beta\left(d\theta_p + X_p\right) \,. \tag{3}$$

The mean value of the disorder operator is defined by

$$\langle \mathcal{D}(X) \rangle = \frac{Z(X)}{Z(0)} \tag{4}$$

and is invariant under the hyper-gauge transformation

$$X \longrightarrow X + d\gamma, \tag{5}$$

with γ a generic 1-form. The hyper-gauge invariance follows by the redefinition of link variables ($\theta \to \theta - \gamma$) and amounts to say that $\langle \mathcal{D}(X) \rangle$ depends only on the 3-form $dX = M$. In fact, by Hodge decomposition, on a convex lattice one can write

$$X = d\alpha + \delta \frac{1}{\Delta} M \tag{6}$$

and the first term on the right-hand side can be always absorbed in a redefinition of θ. Moreover, M turns out to be the dual of the magnetic current density J^M, because the total field strength $G = d\theta + X$ satisfies the following modified Maxwell equations (in presence of an electric current J^E):

$$\delta\,G = J^E \tag{7}$$

$$d\,G = M \quad \text{or} \quad \delta * G = J^M \quad , \tag{8}$$

where by $*G$ we mean the Hodge dual of the form G. Finally, from (8) follows that the magnetic current is identically conserved

$$\delta J^M = 0 \quad . \tag{9}$$

Hence we have that a disorder operator as defined by (4) is unavoidably connected to a magnetic current. In this language a disorder operator describing an open-ended line defect in the statistical system (1) will correspond to a current J^M describing the birth and the evolution of a magnetic monopole. It is possible to parametrize such a conserved magnetic current in a general way as follows:

$$J^M = 2\pi \left[D - \Omega\right] \ . \tag{10}$$

Here Ω is an integer valued 1-form with support on a line (the Dirac string) whose endpoints are the space-time locations of monopoles:

$$\delta\Omega\,(x) = \sum_i q_i\,\delta(x - x_i) \qquad\qquad q_i \in Z\backslash\{0\} \quad . \tag{11}$$

$D_\mu = (0, \vec{D})$ is a superposition of Coulomb-like magnetic fields with flux $\pm q_i$, spreading out at the time slices where the monopoles are located (x_i^0)

$$\vec{D}\,(x_0, \vec{x}) = \pm \frac{q_i}{4\pi}\,\frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}\,\delta(x^0 - x_i^0) \tag{12}$$

in such a way that $\delta D = \delta\Omega$. Moreover, hyper-gauge invariance joint to the compactness of the action implies that the disorder operator does not depend on the shape of the string. In conclusion we see that a magnetic monopole of charge q can be implemented as a defect in the model (1) by a 2-form X whose curvature dX plays the role of a dual magnetic current. For a monopole-antimonopole pair of charge $\pm q$ we have

$$*J^M = d\,X = 2\pi q\,(*D - *\Omega) \qquad X = 2\pi q\,\delta\,\frac{1}{\Delta}\,(*D - *\Omega)\ , \tag{13}$$

where $*\Omega$ is an integer valued 3-form and $*D$ is the 3-form representing the Coulomb field.

B. Mixed order-disorder correlation functions

Performing a Fourier analysis on (3) we obtain

$$Z\,(X) = \sum_{n\,:\,\delta n = 0}\ \prod_{p \subset \Lambda} I_\beta\,(n_p) \quad e^{i(n,X)} \quad . \tag{14}$$

n_p is the integer valued 2-form labelling the Fourier coefficients. With $I_\beta\,(n)$ we indicate the modified Bessel functions of order n evaluated in β (commonly written as $I_n(\beta)$). On the dual form $v = *n$ the constraint $\delta\,n = 0$ becomes $d\,v = 0$ and so we can write $v = d\,A$ and sum over equivalence classes of integer valued 1-forms defined by $[A] = \{A' : d\,A' = d\,A\}$:

$$Z\,(D, \Omega) = \sum_{[A]}\ \prod_{p \subset \Lambda} I_\beta\,(dA_p) \quad e^{i2\pi q(A, D - \Omega)} \quad . \tag{15}$$

Now $e^{i2\pi q(A,\Omega)} = 1$, for integer values of q, and we can conclude that $\langle \mathcal{D}(D, \Omega)\rangle$ actually depends only on (x_i, q_i), once we have fixed the shape of the magnetic field D satisfying $\delta\,D = -\sum_i q_i\,\delta(x - x_i)$.
Now one can introduce ordinary fields, preserving hypergauge invariance of expectation values:

$$\psi_p(D, \Omega) = e^{i[d\theta_p + X_p]} \ . \tag{16}$$

The correlation functions to which the reconstruction theorem applies are then given by

$$S_{n,m}\,(x_1\,q_1, ..., x_n\,q_n;\, p_1......p_m) = \langle \mathcal{D}(x_1\,q_1, ..., x_n\,q_n)\,\psi_{p_1}.......\psi_{p_m}\rangle \tag{17}$$

for $\sum q_i = 0$. Correlation functions with non vanishing total charge are defined by the following limiting procedure for $\sum q_i = q$:

$$S_{n,m} = \lim_{x \to \infty}\, c_q\, S_{n+1,m}\,(x_1, q_1; ......; x_n, q_n; x, -q;\, p_1......p_m) \tag{18}$$

where c_q is a normalization constant. We recall now the reconstruction theorem in the version given in [1].

Theorem 1 *If the set of correlation functions* $\{S_{n,m}\}$ *is*
1) lattice translation invariant;
2) O.S. (reflection) positive;
3) satisfies cluster properties;
then one can reconstruct from $\{S_{n,m}\}$
a) a separeble Hilbert space $\mathcal{H}$ *of physical states;*
b) a vector Ω *of unit norm, the vacuum;*
c) a selfadjoint transfer matrix with norm $\|T\| \leq 1$ *and unitary spatial translation operators* $U_\mu \quad \mu = 1, .., d-1$ *such that*

$$T\,\Omega = U\,\Omega = \Omega\ ;$$

d) Ω *is the unique vector in* $\mathcal{H}$ *invariant under* T *and* U*.*
If moreover the limits (18) vanish, then $\mathcal{H}$ *splits into orthogonal sectors* $\mathcal{H}_q, q \in Z$*, which are the lattice monopole sectors.*

In our case hypotheses 1),2),3) follow from traslation invariance and reflection positivity of the measure defined in the standard way from Wilson action [14]. In particoular one can easily check the reflection positivity of monomials of disorder fields in the dual representation, where they assume a standard form and have support on fixed time slices (because the magnetic field spreads out in fixed time planes).
In the quantum mechanical framework $S_2\,(x, q; y, -q)$ is the amplitude for the creation of a monopole of charge q at the euclidean point x and its annihilation in y. We are now going to show that in the confining phase the two point function $S_2\,(x, q; y, -q)$ is uniformely bounded away from zero, while in the QED phase it vanishes at large euclidean time distances:

$$\lim_{|x-y|\to\infty} S_2\,(x, q\,;\,y, -q) = 0\ . \tag{19}$$

A generalization of these estimates to $S_{n,m}$ implies that in the confining phase there is the so called monopole condensation and in the weak coupling region the physical Hilbert space $\mathcal{H}$ decomposes into orthogonal sectors labelled by total magnetic charge.

II. MONOPOLE CONDENSATION IN THE CONFINING PHASE

In this section we are going to prove monopole condensation in the strong coupling regime: this relies on the fact that given arbitrary $\gamma \in]0, 1[$, there exists a β_γ such that for $\beta \leq \beta_\gamma$ holds

$$S_2\,(x, q\,;\,y, -q) \geq \gamma \qquad \forall\ x, y \in \Lambda\ . \tag{20}$$

In order to prove our statement we shall adopt the following strategy [10]: first, we express the disorder field expectation value in terms of logarithms of the partition functions

$$S_2\,(x, q\,;\,y, -q) = \exp\left[\log Z(D) - \log Z\right] \tag{21}$$

and then we prove that $[\log Z(D) - \log Z]$ is close to 0 uniformely in x and y. The main tool we'll use is a cluster expansion for $[\log Z(D)]$, which we obtain after rearranging $Z(D)$ as the partition function of a polymer gas.

A. The polymer expansion

The first step in our program is the polymer expansion for the (modified) partition function of our system (see e.g. [13,14]). Although it is a standard technique we now briefly present its application to the compact $U(1)$ system with disorder fields. First note that $Z(D, \Omega)$ (15) can be written summing over closed integer valued 2-forms v as follows:

$$Z\,(D, \Omega) = N_\Lambda \sum_{v\,:\,dv=0}\ \prod_{p\subset\Lambda} \tilde{I}_\beta\,(v_p)\quad e^{i2\pi q(A_v, D)}\quad , \tag{22}$$

with

$$N_\Lambda = [I_\beta(0)]^{N_P(\Lambda)} \qquad\qquad \tilde{I}_\beta\,(v_p) = \frac{I_\beta(v_p)}{I_\beta(0)} \quad . \tag{23}$$

In equation (22) A_v is the representative element of the class defined by $dA_v = v$: it is simple to verify that each term in the expansion does not depend on the choice of the representative element A_v. Moreover we have extracted an overall factor rescaling the modified Bessel functions by $I_\beta(0)$: the advantage of this choice is that $\tilde{I}_\beta(0) = 1$ and in our expansion around $v \equiv 0$ we must not carry over tedious factors.
Let us now give some definitions: the support of a k-form $v^{(k)}$ is the following set:

$$\mathrm{supp}\, v^{(k)} = \left\{ x \in \Lambda : x \in c_k \text{ k-cell with } v^{(k)}(c_k) \neq 0 \right\} \; .$$

Let us note that following [10] we think of supp$v^{(k)}$ as a set of points rather then a set of k-cells. In the same way one can define in a natural way the support of a set of k-cells. This permits us to extend to these sets the common definition of connectedness: a set $X \subset \Lambda$ is connected if any two sites in X can be connected by a path of links whose endpoints all lie in X.
Returning to equation (22), in order to recover the equivalent polymer system, we can rearrange it as

$$Z(D) = \sum_{v\,:\,dv=0} k(v) \quad , \tag{24}$$

with (we miss the ininfluent factor N_Λ)

$$k(v) = e^{2\pi q i\,(A_v, D)} \prod_{p \subset \Lambda} \tilde{I}_\beta\,(v_p) \quad . \tag{25}$$

The main idea is to reexpress equations (24) and (25) in terms of closed 2-forms v with connected supports, which will become the supports of the polymers. With this purpose let us recall that it is possible to write [10] $v = \sum_i v_i$ with the property that supp v_i are the connected components of supp v. Moreover one can write $A_v = \sum_i A_{v_i}$ with $dA_{v_i} = v_i$ and these relations allow the following factorisation:

$$k(v) = \prod_i k(v_i) \; . \tag{26}$$

Finally, observing that with the above definition of connected set the condition $dv = 0$ implies $dv_i = 0 \;\; \forall\, i$, we can reorganise the sum in equation (24) as follows:

$$Z(D) = \sum_{n=0}^{\infty} \frac{1}{n\,!} \sum_{X_1 \ldots X_n} \prod_{i=1}^{n} K(X_i, D) \qquad\qquad K(X, D) = \sum_{v:\, supp\, v = X} k(v) \quad . \tag{27}$$

The sum is extended to all finite connected subsets $X_i \subset \Lambda$, with the condition that $X_i \cup X_j$ is disconnected if $i \neq j$. Thus we see that expression (27) has the form of a hard core interaction [13] between polymers X_i of connected support. This allows us to use the techniques developed to deal with these systems in order to give the wanted bound on disorder field expectations. We point out that an expansion of the form given in (27) is common to many statistical (or quantum mechanical in the lattice formulation) systems: the polymer activities $K(X_i, D)$ are the link to the original problem, depending on the form of the starting action. In particular the same expansion is possible for the $U(1)$ gauge model with Villain action [3] and gives monopole condensation in strong coupling [1,4]. The only difference from Wilson model is that the dual representation is gaussian and this simplifies the handling of polymer activities.

B. Bounds on polymer activities

It is a well known result that the main properties (mathematical and physical) of the polymer system can be taken in strict correspondence with the general behavior of the activities, which in turn depends on parameters such as the temperature or the coupling constant. In this subsection we give a bound on $|K(X, D)|$ which is known [13] to be a sufficient condition for the convergence of the Mayer expansion for $\log Z(D)$; moreover it will be of great importance in the estimate of $[\log Z(D) - \log Z]$. We want to show that $\forall\; M > 0 \;\; \exists\; \beta_M$ such that for $\beta < \beta_M$

$$\text{(i)} \quad |K(X,D)| \le e^{-M\,|X|} \qquad\qquad \text{(ii)} \quad |\frac{\partial}{\partial D_b} K(X,D)| \le e^{-M\,|X|} \; . \tag{28}$$

Here $|X|$ stands for the cardinality of the support of the X polymer. The basic tool used in the proof of equation (28) is the following upper bound on the modified (and rescaled) Bessel functions,

$$\tilde{I}_n\,(\beta) \le e^{|n|} \left(\frac{\beta}{2}\right)^{|n|} \qquad\qquad 0 \le \frac{\beta}{2} \le 1 \;, \tag{29}$$

which easily follows from the power expansion [15]:

$$I_n\,(\beta) = \left(\frac{\beta}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\,\Gamma(n+k+1)} \left(\frac{\beta}{2}\right)^{2k} \; .$$

The bound (29) tells us that for small β the series whose n-th term is given by $I_\beta(n)$ is convergent. Now let us sketch the argument that leads to (28). In what follows we assume that $\operatorname{supp} v = X$: from equations (25) and (27) one has

$$|K(X,D)| \le \sum_{v\,:\,dv=0} \prod_{p \subset X} |\tilde{I}_\beta\,(v_p)| \le \sum_{all\ v} \prod_{p \subset X} |\tilde{I}_\beta\,(v_p)| \; . \tag{30}$$

We have added the contributions due to all integer valued 2-forms on X because we have in mind to exploit the convergence property of $\sum I_\beta(n)$. In order to sum over all integer 2-forms with support on X, we first consider each set Y_i of plaquette such that $\operatorname{supp} Y_i = X$ and sum over 2-forms v satisfying $v_p \neq 0 \;\; \forall\, p \in Y_i$. Then we collect the contributions due to all the sets Y_i. Formally we have:

$$|K(X,D)| \le \sum_{Y_i : supp\, Y_i = X} \; \sum_{v : v_p \neq 0} \prod_{p \in Y_i} \left[\tilde{I}_\beta\,(v_p)\right] \; . \tag{31}$$

Now let us focus on the generic term with fixed Y_j: exchanging the sum with the product and using the parity property $I_n\,(\beta) = I_{-n}\,(\beta)$, we obtain for the polymer activity

$$|K(X,D)| \le \sum_{Y_i : supp\, Y_i = X} \; \prod_{p \in Y_i} \left[2 \sum_{n>0} \tilde{I}_\beta\,(n) \right] \; . \tag{32}$$

Thus to each plaquette in Y_i is associated a factor that we extimate replacing $I_\beta(n)$ by the upper bound (29) and summing the resulting series. Noticing that this is a geometric series missing the first term we are left with

$$\prod_{p \in Y_j} \frac{e\beta}{1 - \frac{e\beta}{2}} \le e^{-N_p(Y_j)\,\log(\frac{1}{2e\beta})} \qquad\qquad \text{for} \quad \beta < \frac{1}{e} \;, \tag{33}$$

where $N_p(Y_j)$ is the number of plaquettes in Y_j. From the fact that $\operatorname{supp} Y_j = X$ follows the relation $N_P(Y_j) \ge \frac{1}{4}|X|$. Moreover one can bound the number of Y_j with support on X by $e^{k\,|X|}$ and obtains

$$|K(X,D)| \le e^{-(A_\beta - k)\,|X|} \qquad\qquad A_\beta = \frac{1}{4}\,\log(\frac{1}{2e\beta}) \; . \tag{34}$$

Thus we see that part (i) of (28) is satisfied with $M_\beta = A_\beta - k$. As far as part (ii) of (28) is concerned, it can be obtained showing that there is a function $G(\beta)$ such that holds

$$|\frac{\partial}{\partial D_b} K(X,D)| \le G(\beta) |X|^3 \, e^{-(\frac{|X|}{4} - 1) M_\beta} \equiv F(|X|) \; ; \tag{35}$$

then one finds a constant M'_β such that

$$F(|X|) \le e^{-M'\,|X|} \; . \tag{36}$$

For a proof of equation (35) see appendix A. The proof of (36) can then be obtained by means of elementary analysis.

C. Bound on $S_2\,(x, q\,;\,y, -q)$

For sake of completeness let us now sketch the analysis given in [10] to bound $[\log Z(D) - \log Z]$. Let us start from the definition of the Mayer expansion:

$$\log Z(D) = \sum_{n=0}^{\infty} \frac{1}{n\,!} \sum_{X_1 \ldots X_n} \psi_c(X_1 \ldots X_n) \prod_{i=1}^{n} K(X_i, D)\ . \tag{37}$$

Here $\psi_c(X_1 \ldots X_n)$ is the connected part of the hard core interaction of the polymer system and it is nonvanishing only if $\cup_i X_i$ is a connected set. The bound on the polymer activities allows the convergence of the Mayer expansion [13] and this implies, for example, that correlation functions defined by differentiation of $\log Z(D)$ with respect to D_b share the cluster property.
Following the line of [10] we now define $H(s) = \log Z(sD)$ and observe that $H(s) = H(-s)$ because the measure on equivalence classes $[A]$ is even. Hence we see that

$$\log Z(D) - \log Z(0) = \int_0^1 ds\,(1-s)\,H''(s)\ , \tag{38}$$

where

$$H''(s) = \sum_{b,b'} D(b)D(b')\,m(b,b') \tag{39}$$

$$m(b,b') = \sum_{n=0}^{\infty} \frac{1}{n\,!} \sum_{i,j=1}^{n} \sum_{b \subset B(X_i), b' \subset B(X_j)} \psi_c(X_1 \ldots X_n) \times \left[\prod_{k \neq i,j} K(X_k, sD) \right] \frac{\partial}{\partial D_b} K(X_i, sD)\, \frac{\partial}{\partial D_{b'}} K(X_j, sD)\ . \tag{40}$$

In (40) $B(X)$ denotes the smallest rectangular parallelepiped in Λ which contains X. Moreover, by the exponential bounds on activities and derivatives given in equation (28) follows that

$$\left[\prod_{k \neq i,j} K(X_k, sD) \right] \frac{\partial}{\partial D_b} K(X_i, sD)\, \frac{\partial}{\partial D_{b'}} K(X_j, sD) \le \exp\left[-M \sum_k |X_k| \right]\ . \tag{41}$$

Since $\psi_c(X_1 \ldots X_n) \neq 0$ only if $\cup_i X_i$ is a connected set, we have that $\sum_k |X_k| \ge d(b,b')$ ($d(b,b')$ is the distance between the two links) and so

$$|m(b,b')| \le \exp\left[-\frac{1}{2} M d(b,b') \right] \times \sum_{n=0}^{\infty} \frac{1}{n\,!} \sum_{i,j=1}^{n} \sum_{b \subset B(X_i), b' \subset B(X_j)} \psi_c(X_1 \ldots X_n)\, \exp\left[-\frac{1}{2} M \sum_i |X_i| \right]\ . \tag{42}$$

Now, using the techniques of [13], one can work out from (42) the inequality

$$|m(b,b')| \le \exp\left[-\frac{1}{2} M d(b,b') \right] \times \sum_n n\,\delta^n \sum_{X \supset b} |X|\, \exp\left[-\frac{1}{2} M\,|X| \right]\ , \tag{43}$$

where δ is a constant little as β decreases. For M large enough the sum over n appearing in (43) converges to a constant δ' which again is little as β decreases. Moreover reading the sum in (39) as scalar product between D and mD, we have the following bound on $|H''(s)|$:

$$|H''(s)| = |(D, m\,D)| \le \|D\|_2\, \|m\,D\|_2 \le \|D\|_2^2\, \|m\|_2\ . \tag{44}$$

From properties on matrix norms we have that $\|m\|_2 \le sup_{b'}\,(\sum_b |m(b,b')|)$: using the previous result we obtain

$$|H''(s)| \le \delta'\, \|D\|_2^2 \left[\sum_b e^{-\frac{1}{2} M\, d(b,b')} \right] < \rho(\beta)\ , \tag{45}$$

with $\rho(\beta) \to 0$ as $\beta \to 0$. It is important to point out that the argument works because $\|D\|_2$ is bounded uniformely in x and y. Finally, using (38) and (39) we conclude

$$S_2(x,q\,;\,y-q) > \exp\left[-\frac{1}{2}\,\rho(\beta)\right] \ . \tag{46}$$

This relation implies in particular that $S_1(x,q)$, defined by the limiting procedure in which $y \to \infty$, is nonvanishing. In the language of field theory we can say that the field describing magnetic monopoles aquires a nonvanishing vacuum expectation value. This implies the spontaneous breakdown of the topological symmetry associated to the magnetic charge conservation and signals confinement of electric charge [7].

III. MONOPOLE SECTORS IN THE QED PHASE

In this section we prove the relation (19) for the soliton two point function, showing that for β large enough and $|x-y| \to \infty$ it is possible to find a positive constant $m(\beta)$ such that

$$S_2\,(x,q\,;\,y,-q) \leq e^{-m(\beta)q|x-y|} \ . \tag{47}$$

In order to proof our statement, we use a slight modification of the expansion given in [11,12] and [16]: we reexpress the partition function as a gas of monopole loops, to which apply a renormalization transformation. Estimates on the renormalized loop activities enable us to extract the relevant contribution to $S_2(x,q\,;\,y,-q)$.

A. Expansion of $Z(D,\Omega)$ in interacting monopole loops

From equation (15), defining the modified partition function in dual representation, it is natural to introduce a measure on equivalence classes $[A]$ of integer valued 1-forms given by

$$d\mu\,(A) = \frac{1}{Z}\prod_{p\subset\Lambda} I_\beta\,(dA_p) \qquad\qquad \int_{[A]} d\mu\,(A) = 1\ . \tag{48}$$

The main idea [11] on which is based our construction is the following: we want to introduce a measure $d\mu_{I_\beta}\,(A)$ on R^n (n is the number of links in Λ), which reproduces (48) once we constrain the real variables A_b to integer values and pick them on a gauge slice. Such a measure should enable us to make suitable extimates in the weak coupling region. We can fulfill the first constraint inserting a sum of δ-functions for each link variable A_b, which by Poisson summation gives us the monopole currents; formally we have

$$d\mu\,(A) = \frac{1}{Z}\prod_{b\subset\Lambda}\left[\sum_{q\in Z} e^{i2\pi\,q\,A_b}\right]\,d\mu_{I_\beta}\,(A)\ . \tag{49}$$

Moreover, since we are going to compute expectations values of gauge invariant observables, the only contributions come from conserved currents; hence we can impose

$$\int d\mu_{I_\beta}(A)\,e^{i\,(J,A)} = 0 \qquad\qquad \text{for} \quad \delta\,J \neq 0\ . \tag{50}$$

Actually it is possible to construct a measure with the required properties, taking the limit $\Lambda \to Z^4$ of

$$d\mu_{I_\beta}\,(A) = \frac{1}{N_\Lambda}\prod_{p\subset\Lambda} I_\beta\,(dA)\prod_{b\in\Lambda} dA_b \qquad\qquad A_b \in R\ \ . \tag{51}$$

$I_\beta\,(\phi)$ is a suitable interpolation of the modofied Bessel functions, which has been constructed in [11] for large values of β, and has the properties listed below:

1.

$$I_\beta\,(\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\,e^{\beta\cos\theta}\,e^{i\phi\theta} \qquad\qquad \text{for integer}\ \ \phi\ ;$$

2. $I_\beta(\phi)$ is an *even*, *positive* and *integrable* function on R;

3. $I_\beta(\phi)$ is *analitic* on the strip $|\mathcal{I}\mathrm{m}\,\phi| \le \frac{\beta}{2}$. Moreover it is possible to find a constant c such that

$$|\frac{I_\beta(\phi+ia)}{I_\beta(\phi)}| \le \exp\left[\frac{g(a)}{\beta}\right] \qquad \text{with} \quad 0 \le g(a) \le c\, e^{2\pi|a|} \qquad \text{for} \;\; 1 \le |a| \le \frac{\beta}{2}\ .$$

With these new tools the soliton two point function can be written

$$S_2(x,q\,;\,y,-q) = \int d\,\mu_{I_\beta}(A) e^{i2\pi q(A,(D-\Omega))} \prod_{b\in\Lambda}\left[\sum_{n_b} e^{i2\pi\, n_b A_b}\right]\ . \tag{52}$$

Henceforth, for notational convenience, the factor $2\pi q$ will be absorbed in D and Ω. Our first objective is a suitable rearrangement of the product appearing in (52), which, by the constraint (50), gives the usual coupling of A with external current loops.
Let us start defining a current density ρ as a 1-form on Λ with values in $2\pi Z$. An 1-ensemble $\mathcal{E}$ is a set of current densities $\{\rho\}$ whose supports are disjoint and such that $\mathrm{dist}(\rho,\rho') \ge 2^{\frac{1}{2}}\ \ \forall \rho,\rho' \in \mathcal{E}$. It is useful to collect the above currents in 1-ensembles using the following property [12]:

$$\prod_{b\in\Lambda}\left[\sum_{n_b} e^{i2\pi\, n_b A_b}\right] = \sum_\sigma d_\sigma \prod_{\rho\in\mathcal{E}_\sigma}[1+K(\rho)\,\cos(A,\rho)] \tag{53}$$

The index σ runs over a finite set, each $\mathcal{E}_\sigma$ is an 1-ensemble and $d_\sigma > 0$. Moreover, the bare loop activities satisfy $0 < K(\rho) < e^{k\|\rho\|_1}$, where $\|\rho\|_1$ is the norm given by $\|\rho\|_1 = \sum_{b\in supp\,\rho}|\rho_b|$. The next step is to consider the string Ω as a current density and construct suitable 1-ensembles containing open-ended currents which are obtained by grouping the currents "touching" Ω, in a sense which we'll specify below. We choose Ω (by hyper-gauge invariance) such that $\mathrm{supp}\,\Omega \cap \mathrm{supp}\,D = \oslash$: This means that we are taking Ω with support in the region bounded by the hyperplanes $z^0 = x^0$ and $z^0 = y^0$. The disorder field expectation can be written as follows [11,12,16]:

$$Z(\Omega,D) = \int d\mu_{I_\beta}(A) \sum_\tau c_\tau\ [\cos(A,D-\Omega) + K(\rho^\tau)\,\cos(A,D-\Omega+\rho^\tau)] \times \prod_{\rho\in\mathcal{E}_\tau}[1+K(\rho)\,\cos(A,\rho)]\ . \tag{54}$$

Now, some comments on equation (54): the currents ρ^τ are divergenceless and their support has non vanishing intersection with $N = \{b\in\Lambda\ :\ \mathrm{dist}\,(b,\mathrm{supp}\,(D-\Omega)) \le 1\}$: ρ^τ are the currents touching Ω. c_τ are positive constants and $K(\rho^\tau)$ satisfies $0 < K(\rho^\tau) < e^{k\|\rho^\tau\|_1}$. Moreover $\mathcal{E}_\tau$ is an 1-ensemble of divergenceless currents having vanishing intersection with N. We must point out that $\mathcal{E}_\tau \cup \{\rho^\tau - \Omega\}$ is still an 1-ensemble: we have divided the closed currents from the open-ended $\rho^{\Omega_\tau} = \rho^\tau - \Omega$, which play a peculiar role in the proof of clustering of the soliton correlation function, as we shall see below.

B. Renormalization transformation

Now we make on (54) a transformation that renormalizes the activities of currents and allows us to give the wanted bound on $S_2(x,q\,;\,y,-q)$. The transformation consists in the explicit integration of the factors containing the currents ρ on a suitable subset of $\mathrm{supp}\,\rho$, which we call $\mathcal{B}_\rho$, characterised by the fact that two different links contained in it belong to different plaquettes and

$$\sum_{b\in\mathcal{B}_\rho}|\rho_b| \ge \tilde{c}\,\|\rho\|_1\ . \tag{55}$$

In dimension four the geometric constant $\tilde{c}$ can be fixed to be $\frac{1}{18}$. Such a renormalization is based on the application of the following property. Let us consider a function $G(A)$ which does not depend on A_b for some link $b\in\Lambda$: for arbitrary real a such that $|a| \le \frac{\beta}{2}$ then we have

$$\int e^{iqA_b}\,G(A)\,d\mu_{I_\beta}(A) = e^{-\tilde{E}_\beta(a,q)} \int e^{iqA_b}\left[\prod_{p:b\in\partial p} i_\beta\,(\epsilon(b,p)a,dA)\right] G(A)\,d\mu_{I_\beta}(A)\ , \tag{56}$$

where

$$\tilde{E}_\beta(a,q) = qa - \frac{n_b}{\beta}\, g(a) \qquad \text{and} \qquad i_\beta\,(a,\phi) = \frac{I_\beta\,(\phi + ia)}{I_\beta\,(\phi)}\, e^{-\frac{g(a)}{\beta}} \ . \tag{57}$$

In equation (57) n_b is the number of plaquettes containing b and $\epsilon(b,p)$ is a factor ± 1 which gives the orientation of b in ∂p. The proof of this property is immediate if we make a complex translation on the variable A ($A \to A + ia$) and use the properties of the function $I_\beta(\phi)$.
Now let us focus on the generic term in the sum over the index τ appearing in (54). First, we choose $\mathcal{B}_\rho \subset \operatorname{supp}\rho$ for every $\rho \in \mathcal{E}_\tau$ and $\mathcal{B}_{\rho^\tau} \subset \operatorname{supp}\rho^{\Omega_\tau}$. By the defining property of these subsets and the fact that we are dealing with 1-ensembles, all links selected in this way belong to different plaquettes. After exponentiating the cosines in (54) and decomposing the resulting products, this observation allows us to apply the relation (56) to the resulting terms for all links in $\mathcal{B}_\rho$, $\mathcal{B}_{\rho^\tau}$ and $\operatorname{supp}\Omega$. These integrations produce the transformation of $\cos(A,\rho)$ into a function $c_\beta(A,\rho)$ and renormalize the activities $K(\rho)$ as described by the following relations:

$$Z(\Omega,D) = \int d\mu_{I_\beta}\,(A) \sum_\tau c_\tau\, \left[z(\beta) c_\beta(A, D-\Omega) + z(\beta,\rho^\tau)\, c_\beta(A, D+\rho^{\Omega_\tau})\right] \times \prod_{\rho\in\mathcal{E}_\tau} \left[1 + z(\beta,\rho)\, c_\beta(A,\rho)\right] \ . \tag{58}$$

The renormalized activities are given by

$$\begin{aligned} z(\beta) &= \prod_{b\in\ supp\,\Omega} e^{-\tilde{E}_\beta\,(a,\Omega_b)} \ ; \\ z(\beta,\rho^\tau) &= K(\rho^\tau) \prod_{b\in\mathcal{B}^\tau_\rho} e^{-\tilde{E}_\beta\,\left(a,(D+\rho^{\Omega_\tau})_b\right)} \ ; \\ z(\beta,\rho) &= K(\rho) \prod_{b\in\mathcal{B}_\rho} e^{-\tilde{E}_\beta\,(a,\rho_b)} \ . \end{aligned} \tag{59}$$

The renormalized version of the cosine is $c_\beta\,(A,\rho) = \mathcal{R}e\,[e_\beta\,(A,\rho)]$ where

$$e_\beta\,(A,\rho) = e^{i(A,\rho)} \left[\prod_{p\in T(\mathcal{B}_\rho)} i_\beta\ (\epsilon a, dA(p))\right] \qquad \text{and} \qquad T(\mathcal{B}_\rho) = \{p\in\Lambda \ : \ b\in\partial p \quad b\in\mathcal{B}_\rho\} \ . \tag{60}$$

From the relations given above it is clear that $|c_\beta\,(A,\rho)| \le 1$.

C. Estimates on renormalized activities and cluster property

In order to extract a bound that assures the cluster property we must now choose a suitable value of the parameter a appearing in $\tilde{E}_\beta$. By property 3. of the function $I_\beta\,(\phi)$ follows that

$$e^{-\tilde{E}_\beta\,(a,q)} \le e^{-E_\beta\,(a,q)} \qquad \text{with} \qquad E_\beta\,(a,q) = qa - \frac{n_b}{\beta}\, c\, e^{2\pi|a|} \ . \tag{61}$$

In order to give an upper bound on activities as strong as possible, we take the value af a maximizing $E_\beta(a,q)$ in the domain $|a| \le \frac{\beta}{2}$ and we denote it by a_m. For a *fixed value* of β it turns out that a_m depends on the value of the parameter q, which stands here for the value of the generic current on a link in $\mathcal{B}_\rho$ ($q \to \rho_b$ in equations (59)). We find

$$a^{(1)}_m(q) = \epsilon(q)\, \frac{1}{2\pi} \log\left(\frac{\beta\,|q|}{2\pi n_b c}\right) \ \ \text{if } |q| \le \tilde{q}_\beta; \qquad a^{(2)}_m(q) = \epsilon(q)\, \frac{\beta}{2} \ \ \text{if } |q| \ge \tilde{q}_\beta, \tag{62}$$

and correspondingly

$$E^{(1)}_\beta\,(q) \equiv E_\beta\,(a^{(1)}_m, q) = \frac{|q|}{2\pi} \left[\log\left(\frac{\beta\,|q|}{2\pi n_b c}\right) - 1\right] \ ; \qquad E^{(2)}_\beta\,(q) \equiv E_\beta\,(a^{(2)}_m, q) = |q|\, \frac{\beta}{2} \left(1 - \frac{2 n_b c}{|q|\beta^2}\, e^{\pi\beta}\right) \ . \tag{63}$$

The discriminant value $\tilde{q}$ is defined by $a^{(1)}_m(\tilde{q}) = \frac{\beta}{2}$. The important feature of these current self-energies exctracted by renormalization is that for β large enough both $E^{(1)}_\beta\,(q)$ and $E^{(2)}_\beta\,(q)$ are positive.
With the above choice for the parameter a the renormalized activity of the generic current ρ satisfies

$$z(\beta,\rho) \leq K(\rho) \prod_{b\in\mathcal{B}_\rho} e^{-|\rho_b|\,A(\beta,\rho_b)}\,; \qquad A(\beta,\rho_b) = \begin{cases} \frac{1}{2\pi}\left[\log\left(\frac{\beta\,|\rho_b|}{2\pi n_b c}\right) - 1\right] & \text{for } |\rho_b| \leq \tilde{q}\ , \\ \frac{\beta}{2}\left(1 - \frac{2n_b c}{|\rho_b|\beta^2}\,e^{\pi\beta}\right) & \text{for } |\rho_b| \geq \tilde{q} \end{cases} \tag{64}$$

In both cases $A(\beta,\rho_b)$ is bounded from below by a function which does not depend on ρ_b:

$$A(\beta,\rho_b) \geq \begin{cases} \frac{1}{2\pi}\left[\log\left(\frac{\beta}{12\pi c}\right) - 1\right] = A^{(1)}(\beta) \\ \frac{\beta}{2}\left(1 - \frac{1}{\pi\beta}\right) = A^{(2)}(\beta) \end{cases} \quad . \tag{65}$$

The first bound is obtained using the inequalities $|\rho_b| > 1$ and $n_b \leq 6$; the second one is obtained replacing $|\rho_b|$ with $\tilde{q}$. We must piont out that both $A^{(1)}(\beta)$ and $A^{(2)}(\beta)$ are positive functions increasing with β. If now we define $A(\beta) = \min\{A^{(1)}(\beta), A^{(2)}(\beta)\}$, using the properties of $\mathcal{B}_\rho$ and $K(\rho)$ we can write

$$z(\beta,\rho) \leq K(\rho)\,e^{-\tilde{c}\,A(\beta)\,\|\rho\|_1} \leq e^{-(\tilde{c}\,A(\beta)-k)\,\|\rho\|_1}\ . \tag{66}$$

In particular one can bound the renormalized activity of the string as follows:

$$z(\beta) \leq e^{-\tilde{c}\,A(\beta)\,\|\Omega\|_1} \leq e^{-\tilde{c}\,A(\beta)\,q|x-y|}\ . \tag{67}$$

More delicate is the estimate of $z(\beta,\rho^\tau)$, because of the presence of the Coulomb field D. In fact one has $\mathcal{B}_{\rho^\tau} \subset \mathrm{supp}(\rho^{\Omega_\tau})$ but the generalized current density is $\left(\rho^{\Omega_\tau} + D\right)$. In general for these activities holds

$$z(\beta,\rho^\tau) \leq C(\beta)\,e^{-\tilde{A}(\beta)\,q|x-y|}\ . \tag{68}$$

The starting point to prove (68) is the relation

$$z(\beta,\rho^\tau) \leq K(\rho^\tau) \prod_{b\in\mathcal{B}_{\rho^\tau}} e^{-A(\beta)\,|\rho_b^{\Omega_\tau}+D_b|}\quad . \tag{69}$$

First one can easily see that in the case in which $\mathrm{supp}\,\rho^\tau \cap \mathrm{supp}\,D = \oslash$ one can deal with the current $\rho^{\Omega_\tau} = \rho^\tau - \Omega$ as for the common ρ (see equation (66)) and from the relation $\|\rho^{\Omega_\tau}\| \geq q\,|x-y|$ follows (68) with $C(\beta) = 1$.
The case in which $\mathrm{supp}\,\rho^\tau \cap \mathrm{supp}\,D \neq \oslash$ can be worked out with the following trick: we distinguish the links $b \in \mathcal{B}_{\rho^\tau}$ such that $|D_b| > \frac{1}{2}$ from those such that $|D_b| < \frac{1}{2}$ (in other words we decompose $\mathcal{B}_{\rho^\tau} = \mathcal{B}^<_{\rho^\tau} \cup \mathcal{B}^>_{\rho^\tau}$) obtaining a factorization in equation (69). Noticing then that ρ^τ takes values in $2\pi Z$, for $b \in \mathcal{B}^<_{\rho^\tau}$ we have

$$|\rho_b^{\Omega_\tau} + D_b| = |\rho_b^{\Omega_\tau}|\,|\left(1 + \frac{D_b}{\rho_b^{\Omega_\tau}}\right)| \geq |\rho_b^{\Omega_\tau}|\,\frac{1}{2}\ , \tag{70}$$

by which we reduce the factor involving $\mathcal{B}^<_{\rho^\tau}$ to the standard form (64). The term involving $\mathcal{B}^>_{\rho^\tau}$ will give us the constant $C(\beta)$. In fact after little simple algebra we obtain

$$z(\beta,\rho^\tau) \leq K(\rho^\tau)\,e^{-\frac{1}{2}\,\tilde{c}\,A(\beta)\,\|\rho^{\Omega_\tau}\|_1}\ G(D,\rho^{\Omega_\tau})\ , \qquad \text{with} \qquad G(D,\rho^{\Omega_\tau}) = \prod_{b\in\mathcal{B}^>_{\rho^\tau}} \exp\left[A(\beta)\left(\frac{|\rho_b^{\Omega_\tau}|}{2} - |\rho_b^{\Omega_\tau} + D_b|\right)\right].$$

Finally it is possible to show that $G(D,\rho^{\Omega_\tau})$ is bounded by a constant $C(\beta)$ only dependent on β and the number of links on which $|D| \geq \frac{1}{2}$ (actually this number is a function of q); this completes the proof of equation (68).

Now we have all elements to complete our proof. By the property that $|zc_\beta| \leq |z|$, starting from (58) we can write

$$|S_2(x,q\,;\,y,-q)| \leq \sup_\tau\ \{z(\beta) + z(\beta,\rho^\tau)\} \times \int d\mu_{I_\beta}(A) \sum_\tau c_\tau \prod_{\rho\in\mathcal{E}_\tau} [1 + z(\beta,\rho)\,c_\beta(A,\rho)]\quad . \tag{71}$$

Let us call $\mathcal{R}$ the integral in (71): considering the explicit form of the normalization factor Z, it can be represented as

$$\mathcal{R} = \frac{\sum_\tau A_\tau}{\sum_\tau B_\tau}\ , \tag{72}$$

with

$$A_\tau = c_\tau \int d\mu_{I_\beta}(A) \prod_{\rho \in \mathcal{E}_\tau} \left[1 + z(\beta,\rho)\, c_\beta(A,\rho)\right] \; ; \tag{73}$$

$$B_\tau = c_\tau \int d\mu_{I_\beta}(A) \left[1 + z(\beta,\rho^\tau)\, c_\beta(\rho^\tau, A)\right] \prod_{\rho \in \mathcal{E}_\tau} \left[1 + z(\beta,\rho)\, c_\beta(A,\rho)\right] \; . \tag{74}$$

It is simple to see that for β and $|x-y|$ large enough we have $A_\tau \leq 2\, B_\tau$. In fact choosing suitable values of β and $|x-y|$ we can obtain that $z(\beta,\rho^\tau) \leq \frac{1}{2}$ and $z(\beta,\rho) \leq 1$; thus in order to evaluate B_τ we must integrate in the *positive* measure

$$d\mu_{I_\beta}(A) \prod_{\rho \in \mathcal{E}_\tau} \left[1 + z(\beta,\rho)\, c_\beta(A,\rho)\right]$$

the function $[1 + z(\beta,\rho)\, c_\beta(A,\rho)] \geq \frac{1}{2}$. This observation leads to the conclusion that $\mathcal{R} \leq 2$. Finally using this result in (71) we get the relation

$$|S_2(x,q\,;\,y,-q)| \leq 2\, \sup_\tau \left\{ z(\beta) + z(\beta,\rho^\tau) \right\} \; . \tag{75}$$

Cluster property of the soliton two point function follows then by estimates of equation (67) and (68) on renormalized activities.

IV. CONCLUDING REMARKS

In conclusion we summarize the results: in the weak coupling region the Hilbert space of the reconstructed Lattice Quantum Field Theory splits into orthogonal sectors labelled by magnetic charge. Instead, in the strong coupling region the lattice solitons do condense in the vacuum sector: the symmetry associated to the topological conservation of magnetic charge is spontaneously broken, as signaled by the nonvanishing expectation value of the charged monopole operator. Indeed this is just the criterion for quark confinement proposed by 't Hooft [7].
Moreover our analytical results are in agreement with numerical calculations [17]: these show that the parameter regions in which $S_1 > 0$ and $S_1 = 0$ coincide with the confining phase and QED phase, respectively, and allow to extract information about the behavior of the system at the transition. Our result suggests that the correlation functions of soliton (disorder) fileds are indeed a viable tool for the study, both numerical and analytical, of phase transitions in lattice models that exhibit monopole-like topological excitations (we point out that a slight generalization allows the extension to lattices with non trivial topology). Along this line there is the possibility to study analitically the disorder fields associated with vortices in the $3\,d$ $X\,Y$ model and, although more remote, a generalization to non abelian models.

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APPENDIX A: BOUND ON ACTIVITY'S DERIVATIVES

In this appendix we sketch the argument which leads to equation (35). The proof is based on the following inequality, given in [10]:

$$\|A_v\| \leq \frac{c}{2\pi}\,(v,v)^2 \leq c\, N_p^2\, m_v^4 \; , \tag{A1}$$

where N_p is the number of plaquettes in X and

$$m_v = \max_{p \subset X} |v_p| \quad .$$

The derivative with respect to D_b modifies the expression of $k(v)$ by the moltiplicative factor $2\pi\, A_v(b)$ and thus we have:

$$|\frac{\partial}{\partial D_b} K(X,D)| \leq c N_p^2 \sum_{k>0} k^4 \sum_{[v]:m_v=k} \prod_{p\subset X} \tilde{I}_\beta\,(v_p) \tag{A2}$$

Now we can write

$$\sum_{[v]:m_v=k} \prod_{p\subset X} \tilde{I}_\beta\,(v_p) = \sum_{Y_i:suppY_i=X} \sum_{[v]:m_v=k, v_p\neq 0} \prod_{p\in Y_i} \tilde{I}_\beta\,(v_p)\ . \tag{A3}$$

The 2-forms with $m_v = k$ may be obtained fixing the value of v to $\pm k$ on a plaquette p_M and summing over configurations of integers $n \leq k$ in the remaining $(N_P(Y_i) - 1)$ plaquette. We extract a factor $\frac{\beta}{2}^k$ (coming from $\tilde{I}_\beta(v_{p_M} = k)$) from each term with fixed p_M and notice that the contribution due to all configurations is bounded by $e^{-(N_P(Y_i)-1)\,A_\beta}$. The result is

$$|\frac{\partial}{\partial D_b} K(X,D)| \leq c N_p^2 \sum_{k>0} k^4 \left(\frac{e\beta}{2}\right)^k\ e^{-(|X|-1)\,A_\beta} \sum_{Y_i:suppY_i=X} N_P(Y_i)\ . \tag{A4}$$

The series in k can be estimated by the value of the integral of $x^4 e^{-B\,x}$ on R_+ ($B = \log(\frac{2}{e\beta})$), and this gives $G(\beta)$ apart from factors. Moreover the sum over Y_i is bounded by $2|X|e^{k|X|}$ in such a way that we can write equation (35):

$$|\frac{\partial}{\partial D_b} K(X,D)| \leq G(\beta)|X|^3\, e^{-(|X|-1)M_\beta} \equiv F(|X|)\ . \tag{A5}$$

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